The Matter of Mathematics

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Editor's Note

Russell Howell has co-authored the textbook *Complex Analysis for Mathematics and Engineering*, which is in its sixth edition, and is the co-editor of the HarperOne book *Mathematics Through the Eyes of Faith*. His essay here describes the latest challenges for mathematics and Christian faith. The essay is intended as an invitation. Readers are encouraged to take up one of the insights or challenges, or maybe a related one that was not mentioned, and draft an article (typically about 5,000-8,000 words) that contributes to the conversation. These can be sent to Dr. Howell at howell@westmont.edu. He will send the best essays on to peer review and then we will select from those for publication in a mathematics theme issue of *Perspectives on Science and Christian Faith*. The lead editorial in the December 2013 issue of *PSCF* outlines what the journal looks for in article contributions. For full consideration for inclusion in the theme issue, manuscripts should be received electronically before 30 June 2014.

For those readers who prefer to take a literary approach in sharing their ideas, please submit essays (up to 3,000 words), poetry, fiction, or humour inspired by the invitational essay to emily@asa3.org for possible publication in *God and Nature* magazine.

Looking forward to hearing your perspectives,

James C. Peterson
Editor of *Perspectives on Science and Christian Faith*

1 Introduction

Does faith matter in mathematics? Not according to the Swiss theologian Emil Brunner. In 1937 he suggested a way to view the relationship between various disciplines and the Christian faith. Calling it the *Law of Closeness of Relation*, he commented,

> The nearer anything lies to the center of existence where we are concerned with the whole, that is, with man's relation to God and the being of the person, the greater is the disturbance of rational knowledge by sin; the further anything lies from the center, the less the disturbance is felt, and the less difference there is between knowing as a believer or as an unbeliever. This disturbance reaches its maximum in theology and its minimum in the exact sciences and zero in the sphere of the formal. Hence it is meaningless to speak of a "Christian mathematics."[1]

Thus, Brunner holds a nuanced version of the doctrine of noetic depravity: sin affects the reasoning ability of humans, but does so in varying degrees depending on how "close" the object of reasoning is to their relationship with God. Mathematics, being a purely formal discipline, is beyond the reach of any adverse noetic effects. Christians and non-Christians will therefore come to the same mathematical conclusions, so that, for Brunner, the phrase *Christian mathematics* is an oxymoron.

Of course, on one level Brunner is correct. If one agrees to play the game of mathematics, then one implicitly agrees to follow the rules of the game. Different people following these rules will—Christian or not—agree with the conclusions obtained in the same way that different people will agree that, at a particular stage in a game of chess, white can force checkmate in two moves. In this sense mathematical practice is "world-viewishly" neutral. Moreover, the paradigm for mathematical practice has remained relatively unchanged since Euclid published his masterpiece, *The Elements*, in 300 B.C. That paradigm is to derive results in the context of an axiomatic system.[2]

It would be a mistake, however, to apply Brunner's dictum to all areas of mathematical inquiry. One can be committed to the mathematical game, but also participate in analyzing it (and even criticizing it) from a meta level. In doing so, faith perspectives will surely influence the conclusions one comes to on important questions about mathematics.[3] But is the investigation of such questions really a legitimate part of the mathematical enterprise? At least two reasons can be given for an affirmative answer: (1) such questions are actually taken up at every annual joint meeting of the American Mathematical Society and Mathematical Association of America; (2) historically, such questions have always been investigated by the mathematical community. Indeed, David Hilbert, one of the greatest mathematicians of the twentieth century, chose two topics for discussion in conjunction with the oral defense of his doctoral degree. The first related to electromagnetic resistance. The second was to defend an intriguing proposition: "That the objections to Kant's *a priori* nature of arithmetical judgments are unfounded."[4] Hilbert is credited as being a founder of the school of *formalism*, which insists that axiomatic procedures in mathematics be followed to the letter. It is thus interesting that even those who held a strict view of mathematical practice and meaning saw the investigation of important meta questions relating to mathematics as a legitimate undertaking by mathematicians.

Is there a helpful classification for meta level questions that Christian mathematicians might pursue as they attempt to explore the interaction between their discipline and faith? Arthur Holmes suggests four categories of faith-integration in his well-known book *The Idea of a Christian College*: the foundational, world view, ethical, and attitudinal.[5] The remainder of this essay will look at some developments in mathematics that lead naturally to questions in those categories. It will also suggest (and define) a fifth category for consideration: the *pranalogical*. The ideas presented throughout are by no means meant to be exhaustive, or even representative. It is hoped, though, that they will serve as sufficient triggers for further comment, and for thinking about a wide range of additional meta questions worthy of investigation.

2 Foundational Issues

Holmes states that curricular studies reveal history and philosophy to be common disciplinary areas considered as foundational in higher education.[6] Within the scope of such an education each discipline has historical and philosophical components that have shaped its practices, procedures, and paradigms. Mathematics has a particularly rich tradition. This section delineates a sampling of perspectives that lead to important interactions with the Christian faith.

2.1 Logic

Gottlob Frege thought that all of mathematics is reducible to logic. In 1903 he was about to take a big step in pushing through his program. He had just completed his seminal work, *Grundgesetze der Arithmetik* (the Basic Laws of Arithmetic), volume 2. It contained five axioms that, Frege hoped, would lay the necessary groundwork for all of arithmetic. The axioms were supposedly clear logical statements describing universal truths. If this work succeeded, his goal of producing an unshakable logical foundation would be realized. Unfortunately, just before the book was to be published, Frege received a disturbing letter from Bertrand Russell, who pointed out that Frege's fifth axiom was in conflict with the other four. In other words, Frege's system was inconsistent. It was too late to stop production, so Frege desperately tried to patch things up and inserted a last minute appendix in which he modified his fifth axiom. He also openly explained the situation: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion."[7]

It was subsequently shown that Frege's fix did not work, but the effort to ground mathematics on a rock-solid foundation went on. In 1922 the logicians Ernst Zermelo and Abraham Fraenkel produced a collection of axioms that, together with another axiom called the axiom of choice, serves as the basis for a large portion of mathematics (the theory of sets, which can model what one normally thinks of as arithmetic). This axiom set is still in use today, and is referred to as ZFC. Depending on its formulation, ZFC amounts to about ten axioms.[8]

Why use *these* axioms? As we'll see in a moment, there is not absolute agreement that ZFC is appropriate, but most mathematicians will give at least two reasons for adopting them: (1) the axioms ring true (*i.e.*, they seem worthy of belief);[9] (2) they produce the desired results. The second condition is important. An axiom set that yields unsatisfactory results is not worth much. But this situation raises an interesting question: what renders these results as "desired"? Is it that they conform to commonly shared empirical experiences, or are they independent ontological entities that mathematicians nevertheless somehow sense? If the former, do different people possibly mean slightly different things when they refer to, say, the number five? If the latter, where are these entities located? In God's mind? Section 2.2 briefly explores some of these questions.

Logical Disagreements

One of the disputes regarding the axioms of ZFC arises over the "C" in the acronym, which refers to the axiom of choice. Loosely speaking this axiom stipulates that, given any collection of non-empty bins, it is possible to select one item from each of them. There is no disagreement among mathematicians over the use of this axiom unless the collection of bins is infinite. Even then, there would not be disagreement if, in a specific instance, there were a specified rule or procedure for the selection. For example, if it were known that the bins consisted of positive integers, one could stipulate—even if some bins had infinitely many positive integers—that the smallest integer is to be chosen from each. If, on the other hand, the only knowledge about the bins were that they contained integers (positive or negative), then no constructive procedure could be stipulated ahead of time that would yield a selection. Those accepting the axiom of choice could nevertheless use it to produce a hypothetical selection; those rejecting it would not be able to do so.

Logic and God's Nature

Those who insist that constructive procedures be available in the setting just described likely belong to a school of mathematics known as *Intuitionism*. In general intuitionists deny that there is any external reality to mathematical objects. Rather, mathematical results are only established by human mental constructions. For them, a mathematical result cannot be established by refuting the claim that the result is false; it must be positively proven within the framework of acceptable intuitionistic assumptions. Thus, intuitionists do not subscribe to the law of excluded middle, which states that, for any proposition P either P is true or not-P is true.[10] Intuitionists do subscribe to the law of non-contradiction, which states that, for any proposition P, it cannot be the case that P and not-P both hold.[11]

Intuitionism grew out of objections to results that arose in part from the axiom of choice. Its chief proponent was Luitzen Brouwer (1881–1966), who strongly objected to the seeming paradoxes of Georg Cantor's theory of infinite sets. Section 6 discusses some of these paradoxes. For now we ask a faith-based logical question: Can logical laws be biblically grounded? For example, might 2 Timothy 2:13 ("If we are faithless, He remains faithful, for He cannot deny Himself") support the law of non-contradiction?[12] What about other laws of logic?. The answer to these questions depends on whom you ask.

On the one hand, Sir Michael Dummett (1925–2011), an advocate for intuitionism and a staunch Catholic, rejected classical logic for purely philosophical reasons. He further claimed that his philosophical stance was not influenced in any way by his religious convictions.[13] On the other hand, John Byl, who opts for mathematical realism, attempts to ground a portion of mathematics—including the law of non-contradiction, the axiom of choice, and notions of a completed infinity—on attributes of God found in the scriptures.[14] More generally, Vern Poythress argues that the entire metaphysics of mathematics only holds together coherently because it is part of God's being.[15]

Logic and Gödel

Mathematicians, of course, want coherence, especially in the axioms that help form the building blocks of their edifice. Unfortunately, the theorems that Kurt Gödel produced in 1931 demonstrate that coherence cannot be guaranteed.[16] To explain in full detail the scope of these theorems would go beyond the purpose of this paper. Even lengthy treatises by well-known scholars have come under attack for inaccuracies or misrepresentations.[17] With that caveat out of the way, however, it will be helpful to supply a very brief sketch of Gödel results, as they have important spin-offs for integrative issues. The results apply to any formal axiomatic system that generates an arithmetic capable of addition and multiplication, such as ZFC.[18] In what follows the phrase *the system* will refer to such an axiomatic system.

Painting with very broad strokes, Gödel created a mechanism for associating a unique number with every well-formed proposition.[19] Thus, if P is a particular proposition of the system it will have a number p associated with it, known as its Gödel number. Gödel then created a proposition G that says, loosely, "The proposition whose Gödel number is g cannot be proved using the results of the system." The remarkable feature about G is that it's Gödel number actually is g. Thus, Gödel found a way to have a self-referential statement without the use of potentially ambiguous indexical terms such as the word *this*. In other words, Gödel created an unambiguous way to formulate a proposition that says, roughly,

G : "This proposition cannot be proved within the system."

Gödel then proved two spectacular results:

1. **Theorem 1**: *Within the system, G can be proved if and only if not-G can be proved.*

   There are two important implications of Theorem 1:

   a. If the system is consistent, then neither G nor not-G can be proved within it.

   b. If the law of excluded middle is allowed, then one of the propositions must be true because they are negations of each other. Thus, if the system is is consistent, it contains at least one proposition (either G or not-G) that is true, but cannot be proved.

Corollary: *If the system is consistent, then G is true.*
This corollary can be made plausible via meta reasoning. The proposition G says, of itself, that it cannot be proved. But if the system is consistent, then, indeed, G cannot be proved, so that G asserts the truth (*i.e.*, G is true).

2. **Theorem 2**: *The system cannot be proved to be consistent using the rules of the system.*
The proof of this theorem is fairly straightforward. Suppose the system could be proved to be consistent. Then by the above corollary we would know that G is true, so we would have effectively proven G. But then by Theorem 1, we would also have proven not-G. Thus, G and not-G could *both* be proved, which means the system is not consistent, a contradiction to our assumption. In other words, the assumption that the system can be proved to be consistent leads to an inconsistency. Recall that *the system* refers to any axiomatic system powerful enough to produce an arithmetic capable of addition and multiplication.

Gödel's results have generated a plethora of specious pronouncements. Following is a sample, whose references are not worth reproducing: "Gödel's theorem tells us that nothing can be known for sure; Gödel's incompleteness theorem shows that it is not possible to prove that an objective reality exists; By equating existence and consciousness, we can apply Gödel's incompleteness theorem to evolution."

Regardless of what these comments actually mean it is worth noting the apparent common misunderstanding that Gödel produced *one* theorem. Perhaps that is a red flag to consider when evaluating various pontifications.

Are there any lessons that can be legitimately drawn from Gödel's work? Minimally, his results undercut anyone who might subscribe to a "hyper-foundationalist" program, that is, a program that sets out to prove (in a Descartes-like manner) *everything* that is true by starting with a finite set of indisputable truths or axioms. Gödel demonstrated that not even all mathematical truths can be so established with such a program.

Logic and Mechanism

The Oxford philosopher John Lucas has generated much discussion as a result of his claim that Gödel's theorems refute mechanism.[20] Briefly, Lucas points to the corollary of Gödel's Theorem 1, given earlier: *if the system is consistent, then G is true.* Now, Gödel demonstrated that the truth of G cannot be established within the context of any formal system, and any computer (and computer program) is an instantiation of a formal system (presumably, of course, capable of addition and multiplication): it operates according to the rules governed by its hardware-software configuration. Thus, no computer can "know" the truth of G. Lucas claims, however, that humans can see that G is true.

Here is where the argument gets interesting. According to the corollary humans can see that G is true, but only if they know that "the system" is consistent. Yet Gödel's second theorem stipulates that a proof of this consistency is impossible. So, then, how is it that humans can know this fact? Lucas has answers to this question, and the well-known physicist Roger Penrose agrees with the conclusion, though perhaps for slightly different reasons.[21] Most people, however, disagree with the reasoning Lucas employs—even those who agree with his conclusion that mechanism is false.

Logic and God

Recently (September, 2013) the scholars Christoph Benzmüller and Bruno Woltzenlogel Paleo drew renewed attention to Gödel's ontological proof of God's existence, which he first gave about ten years after his famous incompleteness theorems.[22] Public interest was also captivated by headlines such as "Computer Scientists 'Prove' God Exists."[23]

Gödel's work is a variation of Anselm's ontological argument, which Anselm introduced in chapter two of his famous *Proslogium*:

> Hence, even the fool is convinced that something exists in the understanding, at least, than which nothing greater can be conceived. For, when he hears of this, he understands it. And whatever is understood, exists in the understanding. And assuredly that, than which nothing greater can be conceived, cannot exist in the understanding alone. For, suppose it exists in the understanding alone: then it can be conceived to exist in reality; which is greater.[24]

Benzmüller and Paleo formulated a version of Gödel's argument into a formal system containing five axioms, three definitions, three theorems, and one corollary. The main conclusion is expressed by Theorem 3: "Necessarily, God exists" (in symbols, $\Box\exists x G(x)$). The axioms can be debated, of course, but the system was verified with the help of mechanical theorem provers.[25]

Logic and Computers

Using computerized theorem provers, or using computers in the assistance of a mathematical proof, remains a controversial issue among mathematicians. The controversy came to a head in 1976 when, at a conference in Toronto, Kenneth Appel and Wolfgang Haken announced that they had, with the help of a computer, produced a proof of the "Four Color Theorem." The theorem states that, given any map, it is possible to color it in such a way that no two adjacent regions (such as countries or states) have the same color. Adjacent, of course, means that the regions in question share a measurable linear distance, and not that they meet only at a point (as do Arizona and Colorado in a map of the United States). The proof involves a branch of mathematics known as graph theory, but it was the computer-assisted bit that caused the stir.

For starters, the program did something that no human could possibly do: it verified the theorem to be true for hundreds of thousands of possible cases. A proof requiring a human to do something like that would at least violate the criterion of surveyability that Ludwig Wittgenstein popularized.[26] At a press conference Appel and Haken were asked several questions about the proof:[27] How do you know that the computer itself works properly? (Answer: We've run the program on different machines and gotten the same results.) How do you know that you've considered all the cases? (Answer: Actually, the other day someone sent us a letter pointing out that we had missed several cases. But we entered those missing cases into the computer program and it still came out correct.)

The first question can actually be broken down into three parts: How do you know the computer hardware behaves as advertised? How do you know the program you created is correct? How do you know the compiler that translates your program into machine language is correct? There are formal methods for verifying computer programs, although compiler verification has been of very limited scope.

The answer given to the second question was a bit disconcerting, but the two original questions give rise to interesting additional queries: Is there a Christian perspective on the role of computers and mathematical proof? Would such a perspective involve giving up a certain standard of certainty, a standard normally associated with traditional (and surveyable!) proofs?

2.2 Ontology

Many people have an intuition that mathematical truths are independent of humans. In the words of Martin Gardner, "If two dinosaurs met two others in a forest clearing, there would have been four dinosaurs there—even though the beasts were too stupid to count and there were no humans around to watch."[28] Additionally, mathematical results seem to remain constant across cultures. The mathematical historian Glen Van Brummelen comments that even pre-modern China, which for all practical purposes was mathematically isolated from the rest of the world, exhibits an impressive array of results shared by other cultures, such as the binomial theorem, the solution of polynomial equations via Horner's method, and Gaussian elimination for the solution of systems of linear equations.[29]

Ontological Options

What accounts for this intuition that is seemingly bolstered by the apparent similarity of shared conclusions? A common belief is that mathematical objects have some type of objectively real status that we can access in some way. An alternate approach is to suggest that our common brain structure generates both the intuition and shared conclusions.

Supporters for both views can be found among thinkers from within and outside the Christian tradition. The physicist Sir Roger Penrose posits the existence of three separate worlds with complex interactions: the physical world, the mental world, and the (Platonic) mathematical world.[30] His views have generated a series of objections and responses.[31] Likewise, the mathematician Alain Connes, who argues for an objective, independent existence of mathematical objects, has debated the biologist Jean-Pierre Changeux, who argues that mathematics is merely a product of neural interactions in the human brain.[32] Problems for people with views similar to Penrose and Connes include determining where the mathematical world is located, and coming up with a way to explain how humans have access to this world.

Ontological Realism

The earliest Christian view supporting an objectively real mathematics that is independent of human thinking is probably due to Augustine, who locates propositions such as "$5 + 7 = 12$" in God's mind.[33] With such a view the ontological question relating to the location of mathematical objects dissolves. Further, the means by which we access these ideas can be explained by our having been created in God's image.

As attractive as it sounds, there are substantial problems with Augustine's view. Mathematical truths seem to be *necessarily* true. If so, is God's freedom impaired by the requirement that he *must* conceive of these mathematical thoughts? Chris Menzel has written in detail on issues like this one.[34] An answer to this question, Menzel states, rests on an appeal to God's nature. To say that God necessarily thinks logical thoughts is only to say that God is rational. He can no more refrain from generating them than he can positively commit a sinful act. He cannot do the latter because he is perfectly good. Likewise, being perfectly rational, he cannot do otherwise than to conceive all possible well-formed logical thoughts.

That appears to be a nice solution, but some Christians take issue with it. Roy Clouser, for instance, puts God's thoughts on a different plane from that of humans: "Whereas creatures can't break the law of non-contradiction because they're subject to it, God's transcendent being can't break that law because it doesn't apply to God's being at all."[35]

Those who are comfortable with the idea of logic as part of God's nature, however, have a more serious issue to address. It relates to the contradiction identified by Bertrand Russell that was mentioned in Section 2.1. Basically, Russell showed that a set being a member of itself is an incoherent notion. But if God knows all mathematical truths, then he presumably can conceive of all possible sets. This conception is tantamount to a set of *all* sets, which would mean that such a set has itself as a member. Menzel gets around this difficulty by appealing to what philosophers call an impredicative definition, which is a definition that generalizes over a totality to which the entity being defined belongs. The upshot is that if S is a collection (*i.e.*, a set) of sets, then the sets in that collection must have been well-formed "before" (in a logical sense) they can be aggregated into the set S. Thus, there can be no "set of all sets." To account for God's seeming omnicience of logical constructs Menzel's model has God collecting these logical entities in a hierarchical type-scheme. This model has been formalized in a theory that includes ZFC, and it is provably consistent relative to ZF. Nevertheless, certain difficulties remain,[36] so more work can profitably be done in this area.

Ontological Nominalism

Problems with mathematical realism have led some thinkers to the view that there are no universals or abstract objects.[37] People belonging to this school are dubbed *Nominalists*, coming from the Latin word *nomen*, meaning *name*. Thus, for Nominalists, mathematical objects have no objectively real status. Sets, numbers, and propositions are simply convenient naming devices humans have devised to describe common experiences or thoughts.

Historically, many important philosophers have held this view (*e.g.*, William of Ockham, John Stuart Mill, and George Berkeley), but there is an important issue for the nominalist to sort out. It is often referred to as the *indispensability argument*, popularized by Hilary Putnam and Willard Quine.[38] In a nutshell the argument points out that mathematics is amazingly applicable to the physical world. One might even say that it is indispensable for science. That being the case, there is good reason to believe in the existence of mathematical entities. It is hard to imagine that something nonexistent in reality can nevertheless apply so well to the physical world.

The nominalist Hartry Field took this point seriously. His response to the indispensability argument is the work *Science without Numbers*. In it he attempts to show that, so far as their applications go, mathematical theories need not refer to objectively real objects. Instead, the theories merely need to be "conservative" in the sense that they must be consistent and satisfy a few other minimal conditions.[39] Field then develops "nominalistic axioms" that he claims are sufficient for doing science. Many mathematicians, when looking at these axioms, are unconvinced by the argument. To them, the theory that Field build up looks like another form of mathematics, and a very abstract form at that.[40]

Ontology and the Continuum Hypothesis

The *continuum hypothesis* is due to the work of Georg Cantor (1845–1919), who was the first mathematician to formalize the concept of infinity. Acting out of obedience to carry out his understanding of God's will, Cantor developed a theory of *transfinite numbers*. It was vigorously opposed by well-known mathematicians such as Leopold Kronecker, who, like Brouwer, was an Intuitionist (see Section 2.1). According to Joseph Dauben,

> Cantor believed that God endowed the transfinite numbers with a reality making them very special. Despite all the opposition and misgivings of mathematicians in Germany and elsewhere, he would never be persuaded that his results could be imperfect. This belief in the absolute and necessary truth of his theory was doubtless an asset, but it also constituted

> for Cantor an imperative of sorts. He could not allow the likes of Kronecker to beat him down, to quiet him forever. He felt a duty to keep on, in the face of all adversity, to bring the insights he had been given as God's messenger to mathematicians everywhere.[41]

Cantor showed that infinite sets can be of different sizes. Two infinite sets are the same size (technically, *cardinality*) if there is a one-to-one correspondence between their elements. Thus, the set of natural numbers ($\mathbb{N} = \{1, 2, 3, \ldots\}$) has the same size as the set of even natural numbers ($2\mathbb{N} = \{2, 4, 6, \ldots\}$) because there is a one-to-one correspondence between the two sets: $n \longleftrightarrow 2n$.

From that standpoint it seems at face value that all infinite sets would be of the same size, but Cantor showed otherwise. Remarkably, the set A of all real numbers between zero and one cannot be put into a one-to-one correspondence with $\mathbb{N}$. Mathematicians use the symbol "aleph-null" ($\aleph_0$) to designate the cardinality of $\mathbb{N}$ and c (for "continuum") to designate the cardinality of A.

The continuum hypothesis (henceforth CH) is the assertion that there is no set whose cardinality is between $\aleph_0$ and c. Cantor spent a great deal of effort trying to show that CH is true. At one point he thought he had a proof, but found an error in it. At another point he thought he had a proof that the hypothesis was false, but again found an error. He died without knowing the answer.

In 1940 Kurt Gödel took a big step in proving CH. He showed that, if ZFC is consistent (ZFC is the axiom set discussed in Section 2.1), then so is the axiom set ZFC + CH. In 1963 the Stanford logician Paul Cohen (1934–2007) took a big step in the other direction. Using a technique known as *forcing* he showed that, if ZFC is consistent, then so is ZFC + ¬CH (*i.e.*, ZFC + the negation of CH).[42] Collectively these results demonstrate that, if ZFC is consistent, then CH can be neither proved nor disproved within that system.

So, the question "Is the continuum hypothesis true or false?" actually has *four* possible answers depending on one's philosophical outlook: (1) Yes, mathematical objects are objectively real entities, so CH must be either true or false, and I think it is true. (2) Yes, CH is either true or false, and I think it is false. (3) Yes, CH is either true or false, but I have no inkling as to what the true situation is. (4) No, mathematical objects are not objectively real entities, so there is no universal truth of the matter. Gödel and Cohen collectively have shown that, at least under ZFC, CH is neither true nor false.

The outlook people have on the above question is a good indicator of their ontological viewpoint. In some concluding remarks about the continuum hypothesis textbook author Steven Lay writes, "Thus the continuum hypothesis is undecidable on the basis of the currently accepted axioms for set theory ... It remains to be seen whether new axioms will be found that will enable future mathematicians finally to settle the issue."[43] The thought that the issue can be "settled" probably reveals the author's realist view of mathematical objects.

3 Worldview Issues

Holmes lists four characteristics that comprise a Christian worldview: (1) holistic and integrational (looking at the "big picture"); (2) exploratory (an endless undertaking because a Christian worldview entails that human finiteness is unlikely to exhaust any subject); (3) pluralistic (because Christians, knowing their fallibility, should welcome a variety of perspectives); (4) confessional or perspectival (a Christian worldview starts with an admixture of beliefs, attitudes, and values).[44]

Some of the topics discussed in the previous section could well qualify as being worldview issues. In what follows we highlight a sampling of additional aspects of mathematics that relate to a Christian worldview.

Unreasonable Effectiveness?

In 1960 the Physicist (and eventual Nobel Laureate[45]) Eugene Wigner published an article that has exerted a considerable amount of influence, especially in the past ten years. He saw no satisfactory explanation for the phenomenal success that mathematics seemed to enjoy in the quantum world. Matrix procedures that had been successful with the hydrogen atom were abstracted and applied to the helium atom. Wigner states that there was no warrant for this move because the calculation rules were meaningless in this new context. Yet, the application turned out to be miraculous: "The miracle occurred ... [when] the calculation of the lowest energy level of helium ... [agreed] with the experimental data within the accuracy of the observations, which is one part in ten million. ... Surely, in this case we 'got something out' of the equations that we did not put in."[46] Wigner cites other examples and finally concludes by saying,

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.[47]

Wigner finally received a response from the mathematical community in 1980. The computer scientist Richard Hamming published an article in *The American Mathematical Monthly* in which he gave four "partial explanations" that could account for the success of mathematics:[48] (1) mathematicians craft postulates that conform to things they already have observed, so the implications of those postulates would naturally bear success; (2) mathematicians deliberately select the kind of mathematics that, ahead of time, seems appropriate for a given situation; (3) science (and by implication mathematics) answers comparatively few problems; (4) evolutionary accounts can explain why human reasoning power is successful.

Hamming concludes by saying that his analysis might account for some of the success of mathematics, but doesn't fully explain it. Given Wigner's experience with the hydrogen-helium story, he would probably take issue with Hamming's second point in any case.

In 2008 the logician and mathematical historian Ivor Grattan-Guinness gave a more thorough response to Wigner. He pays careful attention to how different philosophical schools might view the status of theories: as merely devices for calculation (*e.g.*, some forms of positivism); or as explanatory agents (*e.g.*, some forms of Platonism). He then argues that, for the most part, mathematical theories develop in a cultural context, are influenced by other theories already in place, and arise in conjunction with "worldly demands." Referencing Karl Popper, he indicates that there may be an element in science that is guesswork. Sometimes one "hits the bullseye," and that just might have been Wigner's situation in the early stages of quantum theory development.[49] This approach is not necessarily at odds with that of Thomas Kuhn.[50] While Grattan-Guinness is not especially sympathetic with Kuhn's explanation of the structure of scientific revolutions, he does "... accept his advocacy of the *Gestalt* nature of the change."[51]

Grattan-Guinness may have overstated his case somewhat. For example, no physical phenomena guided the formation of complex analysis—a key tool for Wigner. Nevertheless, his case is a powerful one, and reinforces the danger of the "you can't explain this" attitude that sometimess accompanies the Wigner discussion. It is somewhat reminiscent of "God of the Gaps" theories. A problem with them for Christian apologetics is that, potentially, the gaps that seem to exist with current theories may some day be closed up.

Other attempts to answer Wigner's question from a Christian or theistic perspective are more in line with cosmological "fine-tuning" arguments, some kind of gap/fine-tuning hybrid approach, or an "inference to the best explanation" argument. Mark Steiner agrees with Grattan-Guinness in that he criticizes Wigner for ignoring the failures in science, but nevertheless sees the success of mathematics in science as an argument against naturalism. If guesswork is involved in science it is interesting that, as a grand strategy, the bullseye so often is hit when the method employed rests on mathematical theories that invariably grew out of human aesthetic criteria. As Brian Green observes, "Physicists . . . tend to elevate symmetry principles to a place of prominence by putting them squarely on the pedestal of explanation."[52] Steiner sees this outcome as evidence of some sort of privilege that befalls the human species. It makes the universe appear to be "user-friendly" and thus of an anthropocentric character. And any form of naturalism, for Steiner, is *ipso facto* non anthropocentric.[53]

Aesthetics

What aesthetic principles apply in mathematical theory formation? G. H. Hardy developed several ideas in his book *A Mathematician's Apology*. He states that criteria governing "good" mathematics include economy of expression, depth, unexpectedness, inevitability, and seriousness, qualities that also seem to form standards for good poetry.[54] Two of these standards—inevitability and unexpectedness—seem in conflict: how can something inevitable also be unexpected? In a beautiful mathematical proof, however, there is almost always a clever idea that takes the reader by surprise. The idea often reveals a new insight in a similar way that a brilliant move might reveal an opponent's weakness in a chess match. Then, often with other clever ideas, the proof proceeds to a conclusion that in retrospect is inevitable. A similar line of reasoning might apply to the reading of a beautiful poem. It will contain many phrases or nuances that are delightfully new or unexpected. Yet, at the end—paradoxically—there is a feeling that the prose had to be stated the way it was.

What are some Christian perspectives on mathematical aesthetics? There have been various preliminary approaches,[55] but more work in this area would be welcome.

Chance

In 1998 William Dembski published *The Design Inference*, which is a revision of his Ph.D. dissertation in philosophy for the University of Illinois at Chicago. In it he maps out a mathematical theory for detecting design, and thus can legitimately be considered as a founder of the "Intelligent Design" movement. Essentially, the theory makes use of a "design filter," which operates by asking two questions about phenomena that have no natural law explanations: whether they are statistically very unlikely, and whether they contain independently detectable patterns. If the answer to both questions is yes, then design may be reasonably inferred. Dembski tackles problems such as determining how unlikely something must be to pass the filter's test, and indicates that the general thrust of his approach conforms with what people do all the time in attributing design to things they encounter.[56]

Dembski's work has generated a considerable amount of controversy—not so much relating to his filter *per se*, but in his applications of it. An opponent of standard evolutionary explanations for the emergence of life, he is a leading proponent for allowing the teaching of intelligent design as part of the science curriculum in public schools. Along with others, he cites numerous examples of biological systems that purportedly exhibit design as determined by the filter.[57]

Dealing with randomness is awkward for those who view God as sovereign, and also for those who see the universe as a closed, deterministic system. Recently, however, Christian thinkers such as Keith Ward[58] and David Bartholomew[59] have explored the possibility that God may use chance or randomness in fulfilling his purposes for creation. Bartholomew contrasts his thinking with Dembski in the following way:

> The main thesis of the Intelligent Design movement runs counter to the central argument of this book. Here I am arguing that chance in the world should be seen as *within* the providence of God. That is, chance is a necessary and desirable aspect of natural and social processes which greatly enriches the potentialities of the creation. Many, however, including Sproul, Overman and Dembski, see things in exactly the opposite way. To them, belief in the sovereignty of God requires that God be in total control of every detail and that the presence of chance rules out any possibility of design or of a Designer.[60]

It is not clear that Bartholomew is correct in his description of Dembski's apparent opposition to chance; the main point here, however, is to illustrate two very different approaches to a philosophy of chance that Christian thinkers might take.

The topic of chance has become so important that the Templeton Foundation recently funded a large grant that will facilitate scholars in their thinking about the issue.[61] James Bradley, the project director for this grant, has listed some interesting examples of randomness that may hint at divine providence.[62] To illustrate: (1) The process of diffusion, which involves random molecular motion, delivers nutrients to the approximately ten trillion cells in the human body. Thus, randomness serves a purpose in this instance. (2) Some dynamical systems (*e.g.*, Julia sets) produce stable outcomes from random inputs, and other such examples can be found in genetic algorithms and quantum randomness. Thus, order and randomness in these instances are not mutually exclusive.

Bradley has also written about chance for this journal,[63] and for more general readers.[64] Dillard Faries has also published on the topic in this journal.[65] Any additional output that Christian mathematicians might produce in this important area will be a welcome contribution to worldview issues.

Culture

Mathematics has had a profound influence on human culture. For example, it can be argued that a significant amount of modern philosophy has been driven by ontological and other problems raised by the practice of mathematics. A portion of a work edited by Howell and Bradley traces this influence from a Christian perspective.[66] Vladimir Tasić has produced a 157 page volume focusing on a single issue: how mathematics has influenced Postmodern thought.[67]

Both accounts paint with broad strokes, but the grounds are fertile for Christians expounding on more targeted influences of mathematics, influences of which the general public might not be aware. Recent popularizations of important issues have been well received. For example, the basis for the popular search engine used by Google is a sophisticated mathematical algorithm for determining priorities in displaying results of a query. Articles on this topic have appeared in mathematical journals, of course,[68] but accessible books are now on the market, at least for readers with some degree of mathematical sophistication.[69] Information on the process that Google employs is even available for public consumption on Wikipedia.[70]

4 Ethical Issues

According to Holmes, the values that Christians have will show up—consciously or unconsciously—in their work. In the ethical sphere an important component for integrating a discipline with the Christian faith involves what ethicists term "middle-level" concepts, which are the mediators between the "facts" uncovered by a discipline and the biblical values of justice and love. This section explores some possibilities for ethical integration in mathematics.

Disciplinary Worth

Christian educators do not all share the same degree of freedom in the profession of their disciplines. The latitude endorsed by their guilds in determining appropriate choice of topics and assigned readings varies considerably. In mathematics the curricular expectations at the undergraduate level are fairly narrowly focused. Nevertheless, all disciplines share a common concern: whether the discipline itself is worth pursuing.

Two of the standard responses for the worth of mathematics are the *aesthetic argument* (mathematical theories, like great art, have worth simply because of their beauty), and the *future-value argument* (even if a current mathematical theory has no apparent use, theories of mathematics have—historically—eventually resulted in important practical applications). The increasing specialization of mathematics, however, makes these arguments more difficult to sustain. Often, for some highly technical mathematical results, only a dozen or so people fully understand them. If that is the case the aesthetic and future-value arguments are at least threatened: the value of beautiful things that can only be appreciated by a handful of people can be questioned, and mathematical results must have a certain amount of dissemination if they are to have a reasonable chance of one day finding an application. Michael Veatch has written on this conundrum,[71] and further work from a Christian perspective would be welcome.

Disciplinary Apology

Related to the question of disciplinary worth is the need for Christians to develop an apology for the study of mathematics. The section on aesthetics mentioned an apology by G. H. Hardy. It contains many valuable insights, but was written from a secular perspective and produced prior to World War II. Many changes have occurred since then that would no doubt have influenced Hardy's analysis.[72] This author has produced a short apology from a Christian perspective,[73] but a more substantial contribution would render a valuable service to the Christian community.

Disciplinary Pedagogy

The past several years have seen an explosion in pedagogical ideas. In part, it has been driven by the technological revolution. One hears of discussions about MOOCs (massive online open courses), flipped classrooms, IBL (inquiry based learning) practices, and the like. David Klanderman, who specializes in mathematics education, has written on the influence of constructivism in public education,[74] but additional Christian perspectives are needed in evaluating the ever-increasing approaches to education. What, for example, should a Christian response be to pressing factual observations such as the so-called achievement gap in mathematics between various ethnic and social groups? What middle-level concepts can promulgate the biblical values of justice and love in helping overcome the "stereotype threat" that many identifiable groups experience in the mathematical arena?[75]

Should Ethics Influence Mathematics?

Some may claim that ethical considerations should have no bearing on the practice of mathematics. Vern Poythress argues that such a judgment is self-refuting.[76] To see why, label that statement as "C: ethical considerations should have no bearing on the practice of mathematics." Following that as an axiom if you will, it follows that mathematical practice ought not to be influenced by the ethical claim C, which is a self-refuting statement.

5 Attitudinal Issues

Christian mathematicians (indeed, all Christian thinkers) should exhibit practices and affections that grow out of Christian values. According to Holmes,

> If I were teaching symbolic logic, which is as close as a philosopher comes to mathematics, my Christianity would come through with my attitude and integrity far more than in the actual content of the course. A positive, inquiring attitude and a persistent discipline of time and ability express the value I find in learning because of my theology and my Christian commitment.[77]

Holmes goes on to say that these attitudes should affect more than how Christians pursue truth. Their reverence and love for God should also motivate them towards justice (giving all people what they are due, including God), and a desire to act out in practical ways their conviction that every area in the liberal arts—including mathematics—has to do with God.

David Smith gives some nice illustrations of how such attitudes can be played out in teaching the grammar of a foreign language, a subject that is on a similar plane of abstraction as mathematics. He shows how Christian perspectives can be brought to bear in the choice of assigned writing exercises and dialogues used for classroom practice.[78]

Christian mathematics educators can profitably follow Smith's model. Standard exercises in differential equations, for example, can easily be morphed to model phenomena that relate to issues such as ecology or carbon dating that are ripe for Christian involvement. Certain topics by themselves can also serve as springboards for discussion. For example, Wayne Iba has used his training in artificial intelligence to study the proper way in which software programs should render service.[79] What other creative options are possible for Christian mathematicians?

6 Pranalogical Issues

In addition to the four approaches that Holmes delineates, two gospel narratives collectively suggest a fifth category for integrating faith and learning. They share a common feature in that the principles involved are commended by Jesus for their faith.

Pranalogy Defined

The first one, found in Matthew 15, is the story of the Syrophoenician woman. Her daughter is demon possessed. She begs Jesus for help. In an unusual response Jesus says, "It is not good to take the children's bread and throw it to the dogs." The woman replies, "Yes, Lord; but even the dogs feed on the crumbs that fall from their masters' table." Jesus then says, "O woman, your faith is great; it shall be done for you as you wish."

The second instance is recorded in Matthew 8 and also in Luke 7. It is the story of a Roman soldier whose servant is desperately ill. In Matthew's version he comes to Jesus and says, "Lord, I am not worthy for You to come under my roof, but just say the word, and my servant will be healed. For I also am a man under authority, with soldiers under me; and I say to this one, 'Go!' and he goes, and to another, 'Come!' and he comes, and to my slave, 'Do this!' and he does it." Jesus then says to those around him, "Truly I say to you, I have not found such great faith with anyone in Israel." Then he heals the servant.

In addition to the praise given by Jesus in these accounts there is something else they share in common. The faith of both petitioners came in part from their ability to glean a practical spiritual truth by drawing an analogy from what they had learned by experience. The woman did so from behavior she observed among dogs. The soldier likewise understood what authority is by virtue of his occupation, and applied that knowledge to a trust in the authority that Jesus would have to heal.[80]

This analysis gives rise to the additional category for integrating faith and learning. For lack of a better word it should probably be called the *pranalogical* because it involves a practical application of an analogy gleaned from one's discipline or life experience. We could also speak of a *pranalogy*, a word obtained by combining *pra*ctical and a*nalogy*.

There are several potential pranalogical applications of mathematics that can relate to and even enhance one's Christian faith. Following are some suggestions.

Pranalogical Examples

First, as indicated in Section 2.2, Cantor showed that there are actually different sizes of infinity. If the teacher of this theory draws the proper connections it seems inevitable that, once students see and understand the proof of this result, their notion of God being infinitely wise, infinitely powerful, or infinitely good, takes on a new and richer meaning, a meaning that would not be possible without seeing that proof.

Of course, other applications involving the infinite are possible. The work of Benoit Mandelbrot and others in developing fractal geometry has lead to bizarre sets exhibiting self-similarity and infinite detail.[81] Orbits of points whose starting locations are arbitrarily close together are nevertheless radically different. What pranalogies might Christians meaningfully draw from these ideas?

The second application was brought to light long ago by Bishop George Berkeley. In 1734 he composed an essay entitled *The Analyst; or a Discourse Addressed to an Infidel Mathematician.*[82] It is at once a critique of the foundations of calculus, and a rebuke of those scientists who deride people of faith for believing in "mysteries" such as the Trinity that just don't seem to add up. His work closes with a series of 67 pithy rhetorical queries, one of which is "Whether such Mathematicians as cry out against Mysteries, have ever examined their own Principles?"

In other words, Berkeley asserts that—even in mathematics—there are paradoxes. The foundations of calculus have been shored up since Berkeley's time, but paradoxes nevertheless remain. For example, using the axiom of choice Banach and Tarski were able to show that it is possible to decompose a sphere into only five sections. Then, without distorting any of the sections in any way they can be reassembled into two completely contiguous spheres of identical size to the first.[83] Surely that is both a mystery and a paradox.[84]

Returning briefly to Cantor's work, the following facts, when put together, are also paradoxical: (1) between any two rational numbers there is an irrational number; (2) between any two irrational numbers there is an irrational number; (3) these two sets of numbers have no one-to-one correspondence. Thus, there are infinitely more irrational numbers than rational numbers, though infinitely many of both. If pressed to explain this issue a mathematician might say something like, "Well, that's just how things work when dealing with mysterious concepts like infinity."

Indeed, and if things can get so convoluted in a logically precise, carefully defined system such as mathematics, it should be no surprise when paradoxical ideas arise in the Christian faith. The study of mathematics can thus help cope with these paradoxes.

A Pranalogical Caveat

Developing useful pranalogies from one's field of study can be fruitful, but there lurks an obvious danger. In part, it is a danger that accompanies all analogies, but it is especially prominent in mathematics: it is easy to draw analogies that are careless and trite. A well-known mathematician once remarked that the sensitivity of orbits to initial starting locations that Mandelbrot discovered illustrates how God created freedom. Of course, that argument doesn't hold up. The resulting orbits may be sensitively dependent on their starting locations, and in principle the differences in starting locations may be beyond the capabilities of measurement per the Heisenberg uncertainty principle. Nevertheless, the orbits are still absolutely determined by their starting locations.

Thus, in developing pranalogies one must keep in mind the limits of any model, and in dealing with mysteries ultimately return to Paul's statement in First Corinthians, Chapter 13: "For now we see in a glass darkly, but then face to face; now I know in part, but then I will know fully just as I also have been fully known."

Endnotes

[1](Brunner 1946, p. 383). Similar statements can be found in other writings by Brunner. For example (Brunner 2002).

[2]A modern axiomatic system has five components: undefined terms (the basic syntactical strings); definitions (composed of undefined terms); axioms (the unquestioned assumptions from which results will be derived); propositions or theorems (the results so obtained); and rules of reasoning (the methods by which axioms and previously proved theorems will be combined to produce new results).

[3]The recent book *Mathematics Through the Eyes of Faith* begins by listing ten such questions. For details, see (Bradley and Howell 2011, ch. 1).

[4]See (Reid 1996, pp. 16–17).

[5]See (Holmes 1987) I have altered Holmes' original alphabetical listing to correspond with the order presented in this essay.

[6]For Christian colleges, Holmes lists a third area: theology.

[7](Geach and Black 1985, p. 214)

[8]Technically, two of these axioms are actually axiom schemata, each of which contains infinitely many instances. Going into details about this idea, however, is not necessary for the purposes of this paper.

[9]The word *axiom* comes from the Greek *axios* (ἄξιος), meaning *worthy*.

[10]Note the difference between this statement and the law of *bivalence*, which says that, for any proposition P, either P is true or P is false.

[11]For further details see (Shapiro 2000, ch. 7).

[12]This possibility is suggested in (Bradley and Howell 2011, ch. 10).

[13]Michael Dummett confirmed this fact to me in a private conversation over lunch and tea at Wolfson College, Oxford, in June, 2007.

[14]For details, see (Byl 2004, ch. 14).

[15]See (Poythress 1976)

[16]See (Gödel 1986), which contains Gödel's original 1931 paper in German with an English translation on the facing pages.

[17]In fact, the first edition of a careful popularized account by two respected academicians, see (Nagel and Newman 2001), was criticized by Gödel himself.

[18]The model most mathematicians had in mind at the time of Gödel was probably the axiomatic scheme contained in the (eventual) three volume work, *Principia Mathematica*, authored by Alfred North Whitehead and Bertrand Russell. See, for example, (Whitehead and Russell 1963).

[19]Very loosely, a "well-formed" proposition is one created by properly obeying the syntactical rules of the axiomatic system being used.

[20]See (Lucas 1961), also available online, *e.g.*, `http://users.ox.ac.uk/~jrlucas/mmg.html`.

[21]See (Penrose 1994).

[22]See (Gödel 2001).

[23]See `http://abcnews.go.com/Technology/computer-scientists-prove-god-exists/story?id=20678984`.

[24]Available in a variety of publications, and also online at `http://www.fordham.edu/halsall/basis/anselm-proslogium.asp`.

[25]See (Benzmüller and Paleo 2013).

[26]See (Wittgenstein 1983, Part III). By *surveyability* Wittgenstein means that mathematical proofs should be able to be reproduced in the same manner in which an artist can reproduce a picture.

[27]I was privileged to be at this conference; the reporting of the questions and answers that follow are from memory.

[28]See (Gardener 1997).

[29]For further elaboration see (Howell and Bradley 2001, ch. 2).

[30]For details, see (Penrose 1989) and (Penrose 1994).

[31]For example, (Grush and Churchland 1995) and (Penrose and Hameroff 1995).

[32]See (Changeux and Connes 1995).

[33]For a nice elaboration of Augustine's ideas see (Bradley 2007).

[34]See (Howell and Bradley 2001, ch. 3).

[35]See (Clouser 2005, p. 229).

[36]See (Howell and Bradley 2001, footnote 42, pages 93–94).

[37]Technically, a universal is something that has many instances, such as *redness*, whereas abstract objects, such as numbers and sets, do not exist in the physical world. This distinction will not be elaborated on any further.

[38]See, for example (Putnam 1979, pp. 60–68), or (Quine 1981, pp. 1–23).

[39]See (Field 1980).

[40]Anecdotally, when Field's book first came out, many parents of children who feared mathematics rushed to buy the book because of its title. They hoped it would serve as a "gentler, kinder" introduction to science. They were extremely disappointed, however, when they saw the dense notation!

[41](Dauben 1979, p. 291).

[42](Cohen 1963, pp. 1143–1148).

[43]See (Lay 1983, p. 91).

[44](Holmes 1987, pp. 58–59).

[45]The Nobel prize was awarded three years after Wigner published his essay. According to the Royal Swedish Academy of Sciences, the basis for Wigner's selection was "his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles."

[46](Wigner 1960, p. 9).

[47](Wigner 1960, p. 14).

[48]See (Hamming 1980).

[49](Grattan-Guinness 2008).

[50](Kuhn 1996).

[51](Grattan-Guinness 2004, p. 253).

[52](Green 1999, p. 374)

[53]See (Steiner 1998). A fuller elaboration of the aesthetic aspect can be found in (Howell 2008).

[54](Hardy 1967)

[55]For one example see (Bradley and Howell 2011, ch. 7).

[56]See (Dembski 1998). Dembski also addresses other important issues, such as what it means to have an independently detectable pattern.

[57]See, for instance (Behe 2006).

[58](Ward 1996)

[59](Bartholomew 2008)

[60](Bartholomew 2008, p. 99)

[61]For details, visit `http://www.calvin.edu/mathematics/randomnessproject/rfp.html`, accessed January 2014.

[62]`http://www.calvin.edu/mathematics/randomnessproject/examples.html`

[63](Bradley 2013)

[64]See (Bradley and Howell 2011, ch. 5).

[65]See (Faries 2014).

[66]See (Howell and Bradley 2001, chs. 5–7)

[67](Tasić 2001)

[68]For example, see Michael Rempe's essay in (Howell 2011, pp. 131–137).

[69]For example, see (Langville and Meyer 2012).

[70]See `http://en.wikipedia.org/wiki/PageRank`, accessed January 2014.

[71]See (Howell and Bradley 2001, ch. 8).

[72]Hardy takes great pride, for example, that no applications will ever be found in his area of research. Ironically, his number-theoretic results are now very useful in modern-day encryption systems.

[73]See (Bradley and Howell 2011, ch. 11).

[74]See (Howell and Bradley 2001, ch. 12).

[75]For a detailed account of "stereotype threat" see (Steele 2010).

[76]See (Poythress 1976). He actually argues for the incoherence of the claim that mathematics should not be influenced by religious belief. I have only slightly adopted his argument for ethics.

[77](Holmes 1987, p. 47)

[78](Smith 2006)

[79]See (Iba 2013, pp. 54–69).

[80]Robert Brabenec of Wheaton College drew the conclusion relating to the Roman soldier when he shared the account given by Luke in a chapel address on March 25, 2009.

[81]For details, see (Mandelbrot 2002).

[82]Available from Kessinger Publishing's Rare Reprints collection (Berkeley 2004) and also online: http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.html.

[83]For readers of French, see (Banach and Tarski 1924).

[84]One may object to the claim of paradox on the grounds that the two spheres are each non-measurable sets with respect to Lebesgue measure. But that just pushes the conundrum back one step to the question of how there could be such bizarre things as non-measurable sets in the first place.

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